New Questions for Junior High Number Sense (2003)

This document contains information on some of the new tricks that will be appearing on the 2003 District and State Number Sense tests. Some of these tricks are not new and students familiar with the Texas Math and Science Coaches Association (TMSCA) contests will recognize many of the following topics.

The new tricks are sectioned in the order that they will appear on the test. Some of the new tricks are given implicitly. For the others, the student is encouraged to search for an easy mental math formula or procedure for working the problem.

THE PRODUCT OF FOUR CONSECUTIVE INTEGERS, PLUS ONE. [#20-40] For example, $3 \cdot 4 \cdot 5 \cdot 6 + 1$. To find the solution, one could multiply all four numbers together and add one, however, there is a short-cut. The solution can be found by multiplying the first and last number in the sequence, adding one, and squaring the result. Thus, my example yields $(3 \cdot 6 + 1)^2 = (18 + 1)^2 = 19^2 = 361$. (Verify!)

An advanced problem [#60-80] could be asked by giving the result and asking for the smallest (or largest) of the four consecutive integers. For example, "The product of four consecutive positive integers is 360. Find the smallest of these integers." Since the product of four consecutive integers plus one is always a perfect square, add one to 360 to get 361. Since $361 = 19^2$, we know that the product of the smallest and largest integers, plus one is 19. Algebraically, that gives $x(x + 3) + 1 = 19$. This equation simplifies to $x^2 + 3x - 18 = 0$. Factoring gives $(x + 6)(x - 3) = 0$ and the roots are -6 and 3. Since the question stated that only positive numbers are used, we exclude -6 and the answer is 3.

MULTIPLICATION BY 167. [#40-60] This one is obvious. I'll let you think about the trick to this problem. As a starting point, know that the questions will only ask you to multiply multiples of 6 by 167. Good luck.

FACTORIALS, PERMUTATIONS, AND COMBINATIONS. [#60-80] Questions of these types will consist of simply computing factorials, permutations, and combinations. Examples include 5!, $P(6, 2)$, and $C(7, 3)$.

Factorials: As a basis of learning, everyone should know the factorials from 0 to 10. The standard symbol (!) is used to represent factorials. The factorial of a positive integer N is defined as the product of every positive integer from N down to 1. Therefore,

$$N! = N \cdot (N - 1) \cdot (N - 2) \cdots 3 \cdot 2 \cdot 1.$$

With this, we have $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Permutations: The number of permutations of a set (or group of items) is the number of ways in which the items can be re-ordered when the order of the items is meaningful. If the original set has n elements, the permutation $P(n, k)$ gives the number of ways k elements from the original n elements of the set can be re-arranged into a particular order. This is given by the formula

$$P(n, k) = \frac{n!}{(n - k)!}.$$

Therefore,

$$\begin{aligned} P(6,2) &= \frac{6!}{(6-2)!} \\ &= \frac{6!}{4!} \\ &= \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} \\ &= 6\cdot5 \\ &= 30. \end{aligned}$$

Notice that the short-cut to computing the permutation $P(n, k)$ is to multiply the sequence of k integers, starting with n and working your way down towards 1. From above, note that $P(6, 2) = 6 \cdot 5$. That is the product of 2 integers, starting with 6 and working your way down towards 1. With only 2 integers to consider, you don't get far! Another example,

$$\begin{aligned} P(9,4) &= \underbrace{9\cdot8\cdot7\cdot6}_{\text{4 numbers}} \\ &= 3024 \end{aligned}$$

Combinations: Combinations are similar to permutations, except combinations are *unordered*. Thus, for combinations, the order of the re-arrangements does not matter. Combinations are given by the formula

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

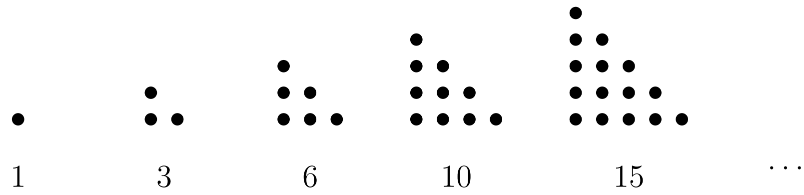
Notice that this is the same as the permutation $P(n, k)$ divided by $k!$. This division by $k!$ removes the multiple counting of sets that have the same elements, but are in a different order. To calculate $C(7, 3)$, begin with the permutation $P(7, 3)$ and divide by 3!.

$$\begin{aligned} C(7,3) &= \frac{P(7,3)}{3!} \\ &= \frac{7\cdot6\cdot5}{3\cdot2\cdot1} \\ &= 7\cdot5 \\ &= 35 \end{aligned}$$

MORE SQUARE ROOTS [#60-80] Some questions on the number sense tests will focus on working with square roots, including reducing square roots completely and multiplying terms with square roots. For example, simplify $\sqrt{48}$. A square root is NOT simplified completely if there is a perfect square (greater than 1) that is a factor of the number inside the square root. This square root ($\sqrt{48}$) is not simplified since $48 = 16 \cdot 3$ and $16 = 4^2$. Simplifying, we get $\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$.

Also, to simplify $(3 - \sqrt{2})(3 + \sqrt{2})$, we must use FOIL. $(3 - \sqrt{2})(3 + \sqrt{2}) = 9 + 3\sqrt{2} - 3\sqrt{2} - \sqrt{4}$. Notice that the middle terms cancel each other out. This happens because the only difference in the two terms is the middle sign. Also, change $\sqrt{4}$ to 2. This leaves us with $9 - 2 = 7$. The answer is 7.

Triangular Numbers [#60-80] The triangular numbers are the numbers that result from building successively larger triangles, as shown in the pattern below.



Let T_n represent the nth triangular number, meaning that T_1 is the first triangular number, T_2 is the second triangular number, and so on. Then, the formula for finding the nth triangular number is $T_n = \frac{n(n+1)}{2}$. For example, "what is the 6th triangular number?" To find the answer, substitute $n = 6$ into the formula. Thus, we have $T_6 = \frac{6(6+1)}{2} = 3(7) = 21$.

To find the sum of two consecutive triangular numbers, $T_{n-1} + T_n$, the result is simply n^2. That is, you take the higher index of the two triangular numbers and square it! For example, "What is the sum of the sixth and seventh triangular numbers?" The answer is $7^2 = 49$ because the indexes of these two triangular numbers are 6 and 7. Since 7 is higher, take 7 and square it.

To find the (positive) difference of two consecutive triangular numbers, $T_n - T_{n-1}$, the answer is simply n. So, "What is the (positive) difference between the eighth and ninth triangular number?" The answer must be 9, since 9 is the higher index.

Finally, the product of two consecutive triangular numbers, $T_{n-1} \cdot T_n$, is found by the formula

$$T_{n-1} \cdot T_n = \frac{1}{4}n^2(n^2 - 1).$$

So, if the question asks, "What is the product of the ninth and tenth triangular numbers?", use the larger index, 10, and the formula:

$$
\begin{aligned}
T_9 \cdot T_{10} &= \frac{1}{4}(10^2)(10^2-1) \\
&= \frac{1}{4}(100)(99) \\
&= (25)(99) \\
&= 2475
\end{aligned}
$$

Practice Questions – The following practice questions cover the above examples and should be used to guide your inquiries into the new types of questions to be asked on the number sense tests.

1. $2 \cdot 3 \cdot 4 \cdot 5 + 1 =$
2. $5 \cdot 6 \cdot 7 \cdot 8 + 1 =$
3. $4! =$
4. $7! =$
5. $P(8, 2) =$
6. $P(5, 3) =$
7. $C(8, 3) =$
8. $C(5, 2) =$
9. Simplify $\sqrt{50}$.
10. Simplify $\sqrt{72}$.
11. $(5 - \sqrt{3})(5 + \sqrt{3}) =$
12. $(8 + \sqrt{10})(8 - \sqrt{10}) =$
13. What is the ninth triangular number?
14. What is the fifth triangular number?
15. What is the sum of the seventh and eighth triangular numbers?
16. What is the sum of the 12th and 13th triangular numbers?
17. What is the positive difference between the seventh and eighth triangular numbers?
18. What is the positive difference between the 19th and 20th triangular numbers?
19. What is the product of the 8th and 9th triangular numbers?
20. What is the product of the 5th and 6th triangular numbers?

This document was prepared by Doug Ray for PSIA, 2002-2003. If you have any questions about the material presented, please email doug@academicmeet.com.

New Questions for Junior High Number Sense (2005)

This document contains information on some of the new tricks that will be appearing on the 2005 District and State Number Sense tests. Some of these tricks are not new and students familiar with the Texas Math and Science Coaches Association (TMSCA) contests will recognize many of the following topics.

The new tricks are sectioned in the order that they will appear on the test. Some of the new tricks are given implicitly. For the others, the student is encouraged to search for an easy mental math formula or procedure for working the problem.

Multiplication by 51. [#20-40] To multiply any even two-digit number by 51, write the 2-digit number on the right side of the answer. Then, divide the number by 2, and write this quotient on the left side of the answer. For example, to multiply 18×51, write 18 on the right side of the answer. Then, $18 \div 2 = 9$. Write 9 on the left side of the answer to produce the product 918.

Multiplication by $16\frac{2}{3}$. [#40-60] I will leave you to figure out the trick to this one. I will, however, let you know that the numbers that the test will ask you to multiply by $16\frac{2}{3}$ will always be multiples of 6.

Fractions in the form $\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n+1}$. [#40-60] To see where the trick for this problem comes from, use algebra to manipulate the fractions by getting a common denominator and adding. So,

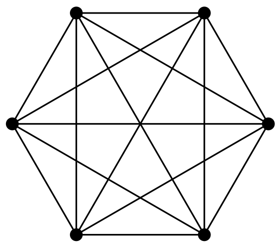
$$\begin{aligned}
\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n+1} &= \frac{1}{n-1}\cdot\frac{n(n+1)}{n(n+1)} - \frac{1}{n}\cdot\frac{(n-1)(n+1)}{(n-1)(n+1)} + \frac{1}{n+1}\cdot\frac{n(n-1)}{n(n-1)} \\
&= \frac{n^2+n-(n^2-1)+n^2-n}{n(n-1)(n+1)} \\
&= \frac{n^2+1}{n(n-1)(n+1)}
\end{aligned}$$

For example, if the problem asks for $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}$, then $n = 3$ (the middle denominator), and by the formula we derived above, the sum is $\frac{3^2+1}{(2)(3)(4)} = \frac{10}{24} = \frac{5}{12}$.

Distinct lines drawn using the vertices of a regular polygon [#60-80] This problem asks you to find the number of distinct (different) lines that can be drawn using the vertices of a regular polygon with n sides as the points on the line. Since the polygon is *regular*, the line between any two vertices will not be part of any other line drawn between other vertices. In other words, using a regular polygon ensures that no three vertices are *collinear*.

Next, to compute the number, you have n vertices from which you must choose 2 points for each line. Therefore, there are $C(n, 2) = n(n - 1)/2$ distinct lines.

For example, to find the number of distinct lines that can be drawn using the vertices of a regular hexagon, use $n = 6$, since a hexagon has 6 sides (and thus, 6 vertices). Then, $C(6, 2) = 6 \cdot (6 - 1)/2 = 15$.



Practice Questions – The following practice questions cover the above examples and should be used to guide your inquiries into the new types of questions to be asked on the number sense tests.

1. $24 \times 51 =$
2. $42 \times 51 =$
3. $32 \times 51 =$
4. $14 \times 51 =$
5. $\frac{1}{4} - \frac{1}{5} + \frac{1}{6} =$
6. $\frac{1}{9} - \frac{1}{10} + \frac{1}{11} =$
7. $\frac{1}{5} - \frac{1}{6} + \frac{1}{7} =$
8. $\frac{1}{12} - \frac{1}{11} + \frac{1}{10} =$
9. $18 \times 16\frac{2}{3} =$
10. $72 \times 16\frac{2}{3} =$
11. $24 \times 16\frac{2}{3} =$
12. $612 \times 16\frac{2}{3} =$
13. How many distinct lines can be drawn using the vertices of a square?
14. How many distinct lines can be drawn using the vertices of a regular pentagon?
15. How many distinct lines can be drawn using the vertices of a regular decagon?
16. How many distinct lines can be drawn using the vertices of a regular 40-gon?

This document was prepared by Doug Ray for PSIA, 2004-2005. If you have any questions about the material presented, please email doug@academicmeet.com.

New Questions for Junior High Number Sense (2007)

This document contains information on some of the new tricks that will be appearing on the 2007 District and State Number Sense tests. Some of these tricks are not new and students familiar with the Texas Math and Science Coaches Association (TMSCA) contests will recognize many of the following topics.

The new tricks are sectioned in the order that they will appear on the test. Some of the new tricks are given implicitly. For the others, the student is encouraged to search for an easy mental math formula, procedure for working the problem, or information on the topic.

Multiplication by $6\frac{1}{4}$ [#20-40] Since $6\frac{1}{4} = \frac{25}{4} = \frac{100}{16}$, the trick to multiplying by $6\frac{1}{4}$ is to divide the number by 16 first, then multiply by 100.

For example, to multiply $48 \times 6\frac{1}{4}$, divide $48 \div 16 = 3$, and then multiply $3 \times 100 = 300$.

Sum of $\frac{r^{n-1}-1}{r^{n-1}} + \frac{r^n-1}{r^n} + \frac{r^{n+1}-1}{r^{n+1}}$ [#40-60] First of all, to recognize this special sum, you should notice the increasing powers of some number in the denominator. These powers will be in order. Next, you should notice that the numerator is just one less than the denominator in each fraction.

To find the sum, consider the following: each fraction can be split up and the sum re-associated.

$$\begin{aligned}\frac{r^{n-1}-1}{r^{n-1}} + \frac{r^n-1}{r^n} + \frac{r^{n+1}-1}{r^{n+1}} &= \left(1-\frac{1}{r^{n-1}}\right)+\left(1-\frac{1}{r^n}\right)+\left(1-\frac{1}{r^{n+1}}\right)\\ &= 3-\left(\frac{1}{r^{n-1}}+\frac{1}{r^n}+\frac{1}{r^{n+1}}\right)\\ &= 3-\left(\frac{r^2+r+1}{r^{n+1}}\right)\end{aligned}$$

Notice that the fraction $\frac{r^2+r+1}{r^{n+1}}$ does not reduce (why?) and that $r^2+r+1 < r^3$ for $r \geq 2$ (is there a better lower bound?). Since you are subtracting the fraction from 3, it will be easy to write the answer as a mixed number.

For example, $\frac{7}{8}+\frac{15}{16}+\frac{31}{32} = 3-\left(\frac{2^2+2+1}{32}\right) = 3-\frac{7}{32} = 2\frac{25}{32}$.

Platonic Solids [#60-80] I will leave this information for you to dig up. The five Platonic Solids are tetrahedron, cube (hexahedron), octahedron, dodecahedron, and icosahedron. You should know the number of vertices, edges, and faces of each.

QUADRATICS [#60-80] The emphasis here is the factorization of a quadratic, the number of real roots (or zeros if written as a function), and the sum or product of the roots.

Much of the information about the roots of a quadratic is known even without knowing the roots themselves and is given by the *discriminant*. The discriminant of the quadratic $ax^2 + bx + c = 0$ is $D = b^2 - 4ac$. Notice that this is just part of the Quadratic Formula. If $D > 0$, then the quadratic has 2 real distinct roots. If $D = 0$, then the quadratic has 1 real double root. If $D < 0$, then the quadratic does not have any real roots.

Finally, for the quadratic $ax^2 + bx + c = 0$, the sum of the roots is $-b/a$ and the product of the roots is c/a.

PRACTICE QUESTIONS – The following practice questions cover the above examples and should be used to guide your inquiries into the new types of questions to be asked on the number sense tests.

1. $6\frac{1}{4} \times 16 =$
2. $96 \times 6\frac{1}{4} =$
3. $32 \times 6\frac{1}{4} =$
4. $4.8 \times 6.25 =$
5. $176 \times 6.25 =$
6. $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} =$
7. $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} =$
8. $\frac{4}{5} + \frac{24}{25} + \frac{124}{125} =$
9. $\frac{80}{81} + \frac{26}{27} + \frac{8}{9} =$
10. $\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} =$
11. How many edges does a cube have?
12. How many faces does a dodecahedron have?
13. How many vertices does a tetrahedron have?
14. A Platonic solid has 8 vertices. How many edges does it have?
15. A Platonic solid has twice as many edges as vertices. How many faces does it have?
16. How many distinct real roots does the quadratic $x^2 + 8x + 15 = 0$ have?
17. Find the discriminant of $9x^2 + x - 3 = 0$.
18. Find the sum of the roots of $8x^2 + 24x - 35 = 0$.
19. The product of the roots of $3x^2 + 17x + 18 = 0$ is
20. If $x^2 - 9x + 20$ factors as $(x - A)(x - B)$ with $A < B$, find A.

This document was prepared by Doug Ray for PSIA, 2006-2007. If you have any questions about the material presented, please email doug@academicmeet.com.

New Questions for Junior High Number Sense (2009)

This document contains information on some of the new tricks that will be appearing on the 2009 District and State Number Sense tests. Some of these tricks are not new and students familiar with the Texas Math and Science Coaches Association (TMSCA) contests will recognize many of the following topics.

The new tricks are sectioned in the order that they will appear on the test. Some of the new tricks are given implicitly. For the others, the student is encouraged to search for an easy mental math formula, procedure for working the problem, or information on the topic.

Multiplication by 99 [#20-40]

The trick to multiplying a two-digit number by 99 is to recognize $99 = 100 - 1$. Thus, write down 1 less than the number for the left digits in the answer, and subtract the two-digit number from 100 to get the right digits.

For example, to compute 23×99, write down $23 - 1 = 22$ on the left and $100 - 23 = 77$ on the right. The answer is 2277.

Sum of $\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}$ [#40-60]

This trick involves adding fractions where the denominators are the first three powers of a number n. To see where the trick comes from, we use the following:

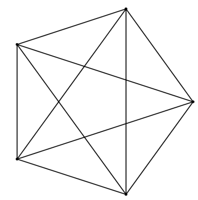
$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} = \left(\frac{1}{n} \cdot \frac{n^2}{n^2}\right) + \left(\frac{1}{n^2} \cdot \frac{n}{n}\right) + \left(\frac{1}{n^3}\right) \qquad \text{The common denominator is } n^3$$

$$= \frac{n^2 + n + 1}{n^3} \qquad \text{Combine the fractions}$$

This equation tells us that the numerator of the sum is found by adding the first two powers of n (listed in the denominators of the problem) and also adding 1. The denominator is simply n^3.

For example, to find the sum $\frac{1}{5} + \frac{1}{25} + \frac{1}{125}$, recognize that $5^2 = 25$ and $5^3 = 125$. This matches our pattern. To find the sum, take $25 + 5 + 1 = 31$ for the numerator and 125 for the denominator. The answer is $\frac{31}{125}$.

NUMBER OF DISTINCT DIAGONALS IN A REGULAR POLYGON
[#60-80] A regular polygon with n sides has $\frac{n(n-3)}{2}$ distinct diagonals. Here, "distinct" means that we count each diagonal only once. Thus, the diagonal AF is the same as diagonal FA and is counted once.



To find the number of distinct diagonals of a regular pentagon, use the formula with $n = 5$: $D = \frac{5(5-3)}{2} = 5$.

SUM OF FIRST n CUBE NUMBERS: $1^3 + 2^3 + 3^3 + \ldots + n^3$ [#60-80]
The sum of $1^3 + 2^3 + 3^3 + \ldots + n^3$ is $\left[\frac{n(n+1)}{2}\right]^2$. You may recall that $\frac{n(n+1)}{2}$ is the nth triangular number. Thus, the sum of the first n cube numbers is the square of the nth triangular number.

For example, to find $1^3 + 2^3 + 3^3 + 4^3 + 5^3$, find the 5th triangular number: $\frac{5(6)}{2} = 15$. Square this number: $15^2 = 225$. The answer is 225.

PRACTICE QUESTIONS – The following practice questions cover the above examples and should be used to guide your inquiries into the new types of questions to be asked on the number sense tests.

1. $99 \times 45 =$ ________________.
2. $81 \times 99 =$ ________________.
3. $63 \times 99 =$ ________________.
4. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} =$ ________________.
5. $\frac{1}{7} + \frac{1}{49} + \frac{1}{343} =$ ________________.
6. $\frac{1}{9} + \frac{1}{81} + \frac{1}{729} =$ ________________.
7. How many distinct diagonals does a regular hexagon have? ________________.
8. How many distinct diagonals does a square have? ________________.
9. A regular n-gon has 90 distinct diagonals. Find n. ________________.
10. $1^3 + 2^3 + 3^3 + \ldots + 6^3 =$ ________________.
11. $1^3 + 2^3 + 3^3 + \ldots + 9^3 =$ ________________.
12. $\sqrt{1^3 + 2^3 + 3^3 + \ldots + 20^3} =$ ________________.

This document was prepared by Doug Ray for PSIA, 2008-2010. If you have any questions about the material presented, please email `doug@academicmeet.com`.